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Theory of q-Deformed Forms. II. q-Deformed Differential Forms and q-Deformed Hamilton Equation

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In this paper we introduce the q-deformed differential forms and quantum-algebra-valued q-deformed forms. We use these to obtain the q-inner derivative and investigate its properties. As a physical application we discuss the q-deformed Hamilton equation.

1. INTRODUCTION

Quantum groups provide a concrete example of noncommutative differential geometry (Connes, 1986). The idea of the quantum plane was first introduced by Manin (1988, 1989). The application of noncommutative differential geometry to quantum matrix groups was made by Woronowicz (1987, 1989). Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's quantum plane. They developed a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$, so-called $GL_q(n)$. Much subsequent work has been done in this direction (Schmidke *et al.*, 1989; Schirrmacher, 1991a,b; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Ubriaco, 1992; Giler *et al.*, 1991, 1992; Lukierski *et al.*, 1991; Lukierski and Nowicki, 1992; Castellani, 1992; Chaichian and Demichev, 1992; Chung, n.d.-a,b).

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In part I (Chung *et al.*, 1996) of this series we proved associativity of the q-deformed wedge product and showed that the q-deformed wedge product satisfies a particular commutation relation.

In this paper we introduce the q-analogue of differential forms and discuss their properties. We define the q-deformed differential forms, quantum-algebra-valued q-deformed forms, and the q-deformed inner derivative. We use these results to obtain the q-deformed Hamilton equation.

2. q-DEFORMED DIFFERENTIAL FORMS

In this section we introduce the q-deformed differential forms and investigate their properties. We define the q-deformed differential forms in terms of the q-deformed wedge product as follows:

$$dx^i \wedge_q dx^j = (-q) dx^j \wedge_q dx^i \qquad (i > j)$$
$$dx^i \wedge_q dx^i = 0 \qquad (i, j = 1, 2, \ldots, n) \tag{1}$$

where the q-deformed wedge product $\wedge_q$ reduces to the usual wedge product when q goes to 1. The relation (1) can be written in the form

$$dx^i \wedge_q dx^j = (-q)^{P(ij)} dx^j \wedge_q dx^i \tag{2}$$

where the symbol $P(ij)$ is defined as

$$P(ij) = 1 \qquad (i > j)$$
$$P(ij) = 0 \qquad (i = j)$$
$$P(ij) = -1 \qquad (i < j)$$

Let V be a vector space and $\Lambda_q^p V$ a space of the q-deformed p-forms over V. We then introduce the basis dx^I of $\Lambda_q^p V$ as follows:

$$dx^I = dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p} \in \Lambda_q^p V \qquad (i_1 < i_2 < \cdots < i_p) \tag{3}$$

Then, for the q-deformed p-form basis

$$dx^I = dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p} \in \Lambda_q^p V \qquad (i_1 < i_2 < \cdots < i_p) \tag{4}$$

and the q-deformed l-form basis

$$dx^J = dx^{j_1} \wedge_q \cdots \wedge_q dx^{j_l} \in \Lambda_q^l V \qquad (j_1 < j_2 < \cdots < j_l) \tag{5}$$

the following commutation relation exists:

$$dx^I \wedge_q dx^J = E^{IJ}_{JI} dx^J \wedge_q dx^I \tag{6}$$

where

$$E^{IJ}_{JI} = \frac{E_{IJ}}{E_{JI}}$$

$$= \frac{E_{i_1 \cdots i_p j_1 \cdots j_l}}{E_{j_1 \cdots j_l i_1 \cdots i_p}}$$

$$= (-q)^{\Sigma_{m=1}^{p} \Sigma_{n=1}^{l} P(i_m, j_n)} \tag{7}$$

Here the q-deformed Levi-Civita symbol $E_{i_1 \cdots i_N}$ is defined as (Chung *et al.*, n.d.)

$$E_{12 \cdots N} = 1$$

$$E_{\cdots ij \cdots} = (-q) E_{\cdots ji \cdots} \qquad \text{for} \quad i > j$$

The last relation of (7) holds because

$$E^{IJ}_{JI} = \frac{E_{i_1 \cdots i_p j_1 \cdots j_l}}{E_{j_1 \cdots j_l i_1 \cdots i_p}}$$

$$= \frac{(-q)^{\Sigma_{m=1}^{p} \Sigma_{n=1}^{l} P(i_m, j_n)} E_{j_1 \cdots j_l i_1 \cdots i_p}}{E_{j_1 \cdots j_l i_1 \cdots i_p}}$$

$$= (-q)^{\Sigma_{m=1}^{p} \Sigma_{n=1}^{l} P(i_m, j_n)}$$

which shows that for the $q \to 1$ limit this becomes $(-1)^{pl}$. Using this, we can prove the relation (6),

$$dx^I \wedge_q dx^J$$

$$= (dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}) \wedge_q (dx^{j_1} \wedge_q \cdots \wedge_q dx^{j_l})$$

$$= (-q)^{\Sigma_{m=1}^{p} P(i_m, j_1)} dx^{j_1} \wedge_q dx^I \wedge_q (dx^{j_2} \wedge_q \cdots \wedge_q dx^{j_l})$$

$$= (-q)^{\Sigma_{m=1}^{p} \Sigma_{n=1}^{l} P(i_m, j_n)} dx^J \wedge_q dx^I$$

Then an arbitrary q-deformed p-form α is given by the linear combination of the q-deformed p-form basis $dx^I \in \Lambda_q^p V$:

$$\alpha = \sum_I \alpha_I dx^I \in \Lambda_q^p V \tag{8}$$

$$= \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p} \tag{9}$$

The q-deformed p-form α is also written as

$$\alpha = \frac{1}{[p]!} \sum_{i_1, \ldots, i_p} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$

where

$$\alpha_{\cdots ij \cdots} = (-q^{-2})^{P(i,j)} \alpha_{\cdots ji \cdots}$$

and

$$[p] = \frac{1 - q^{-p}}{1 - q^{-1}}$$

Using the commutation relation between the q-deformed p-form basis and the q-deformed l-form basis, we have the commutation relation between an arbitrary q-deformed p-form α

$$\alpha = \sum_I \alpha_I dx^I \in \Lambda_q^p V \tag{10}$$

and an arbitrary l-form β

$$\beta = \sum_J \beta_J dx^J \in \Lambda_q^l V \tag{11}$$

This is written as

$$\alpha \wedge_q \beta = (-q^{-2})^{pl} \hat{Q}(\beta \wedge_q \alpha) \tag{12}$$

where $\hat{Q}$ is a operator transforming the q-wedge product $\wedge_q$ into the q^{-1}-wedge product ($\wedge_{q^{-1}}$), that is,

$$\hat{Q}(\wedge_q) = \wedge_{q^{-1}} \tag{13}$$

Equation (12) is easily proved in a same way as given in Chung *et al.* (1996). In order to obtain the q-deformed Leibniz rule, we have the identity

$$dx^a \wedge_q \alpha = \alpha_{*a} \wedge_q dx^a \tag{14}$$

where α_{*a} is defined as

$$\alpha_{*a} = \sum_{i_1 < i_2 \cdots < i_N} (-q)^{\rho(a, i_1, i_2, \cdots, i_N)} \alpha_{i_1 i_2 \cdots i_N} \times dx^{i_1} \wedge_q dx^{i_2} \wedge_q \cdots \wedge_q dx^{i_N} \tag{15}$$

and

$$\rho(a, i_1 \cdots i_p) = -\sum_{k=1}^{N} P(a, i_k) \tag{16}$$

Equation (16) is easily proved:

$$\begin{aligned} dx^a \wedge_q \sum_I \alpha_I dx^I &= \sum_I (-q)^{\sum_{m=1}^{p} P(a, i_m)} dx^I \wedge_q dx^a \\ &= \alpha_{*a} \wedge_q dx^a \end{aligned}$$

where

$$\alpha_{*a} = \sum_I \alpha_I (-q)^{\sum_{m=1}^{p} P(a,i_m)} dx^I$$

In particular for $\alpha \in \Lambda_q^0$ we get

$$dx^a \wedge_q \alpha = \alpha \wedge_q dx^a$$

which means

$$\alpha_{*a} = \alpha$$

Using the relation (16), we obtain the q-deformed Leibniz rule for the q-deformed forms,

$$d(\alpha \wedge_q \beta) = d\alpha \wedge_q \beta + \sum_a \alpha_{*a} \wedge_q dx^a \partial_a \beta \tag{17}$$

The derivation of eq. (17) is then given by

$$\begin{aligned}
&d(\alpha \wedge_q \beta) \\
&\quad = d \sum_{I,J} \alpha_I dx^I \wedge_q \beta_J dx^J \\
&\quad = \sum_{I,J} d(\alpha_I \beta_J) \wedge_q dx^I \wedge_q dx^J \\
&\quad = \sum_{I,J} (d\alpha_I \beta_J + \alpha_I d\beta_J) \wedge_q dx^I \wedge_q dx^J \\
&\quad = d\alpha \wedge_q \beta + \sum_{I,J} \alpha_I \partial_a \beta_J dx^a \wedge_q dx^I \wedge_q dx^J \\
&\quad = d\alpha \wedge_q \beta + \sum_{I,J} \alpha_I \partial_a \beta_J (-q)^{\sum_{m=1}^{p} P(a,i_m)} dx^I \wedge_q dx^a \wedge_q dx^J \\
&\quad = d\alpha \wedge_q \beta + \sum_{I,J} \alpha_I (-q)^{\sum_{m=1}^{p} P(a,i_m)} dx^I \wedge_q \partial_a \beta_J dx^a \wedge_q dx^J \\
&\quad = d\alpha \wedge_q \beta + \sum_a \alpha_{*a} \wedge_q dx^a \partial_a \wedge_q \beta
\end{aligned}$$

In particular, for $\alpha \in \Lambda_q^0$ we get

$$\begin{aligned}
&d(\alpha \wedge_q \beta) \\
&\quad = d\alpha \wedge_q \beta + \sum_a \alpha \wedge_q dx^a \partial_a \beta \\
&\quad = d\alpha \wedge_q \beta + \alpha \wedge_q d\beta
\end{aligned}$$

Then for $\phi \in La_q^p$ we have the following q-deformed Poincaré lemma:

$$d^2\phi = 0 \qquad \text{whenever} \quad \partial_i \partial_j = q^{P(j,i)} \partial_j \partial_i$$

By definition we get

$$d\phi = d\phi_{i_1\cdots i_p} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$
$$= \partial_a \phi_{i_1\cdots i_p} dx^a \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$

Then we have

$$d^2\phi = \partial_b \partial_a \phi_{i_1\cdots i_p} dx^b \wedge_q dx^a \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$
$$= (-q)^{P(b,a)} \partial_b \partial_a \phi_{i_1\cdots i_p} dx^a \wedge_q dx^b \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$

where we used

$$dx^b \wedge_q dx^a = (-q)^{P(b,a)} dx^a \wedge_q dx^b$$

If

$$\partial_b \partial_a = q^{P(a,b)} \partial_a \partial_b$$

then we obtain

$$d^2\phi = -\partial_a \partial_b \phi_{i_1\cdots i_p} dx^a \wedge_q dx^b \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$
$$= -d^2\phi$$

which means that

$$d^2\phi = 0$$

3. q-DEFORMED QUANTUM ALGEBRA-VALUED FORMS

The purpose of this section is to generalize q-deformed forms to q-deformed forms with values in quantum algebra g. Let $\Lambda_q^p(V, g)$ be a space of q-deformed p-forms over V with values in quantum algebra g. Then the product of an element belonging to $\Lambda_q^p(V, g)$ and an element belonging to $\Lambda_q^r(V, g)$ is defined in terms of the qummutator

$$[\phi, \psi]_q = \sum_{a,b} \phi_a \wedge_q \psi_b [T_a, T_b]_q \tag{18}$$

where

$$\phi = \sum_a \phi_a T_a, \qquad \in \Lambda_q^p(V, g) \tag{19}$$

$$\psi = \sum_b \psi_b T_b, \qquad \in \Lambda_q^r(V, g) \tag{20}$$

and the qummutator is defined as

$$[A, B]_q = qAB - q^{-1}BA \tag{21}$$

Here T_a and T_b mean the generators of the quantum algebra g satisfying the qummutation relation

$$[T_a, T_b]_q = f_{abc}T_c \tag{22}$$

where f_{abc} is a structure constant. This type of quantum algebra is usually called a Cartesian version of quantum algebra. Then the graded anticommutativity is deformed into

$$[\phi, \psi]_q = -(-q^{-2})^{pr}\hat{Q}([\psi, \phi]_{q^{-1}}) \tag{23}$$

The Jacobi identity is deformed as follows:

$$\begin{aligned}&[[\phi, \psi]_q, \chi]_q + (-q^{-2})^{-p(r+s)}\hat{Q}([[\psi, \chi]_q, \phi]_{q^{-1}}) \\ &\quad + (-q^{-2})^{-s(p+r)}\hat{Q}([[\chi, \phi], \psi]) = 0\end{aligned} \tag{24}$$

where we used the q-deformed Jacobi identity for the basis of g,

$$[[T_a, T_b]_q, T_c]_q + [[T_b, T_c]_q, T_a]_{q^{-1}} + [[T_c, T_a], T_b] = 0 \tag{25}$$

and ϕ, ψ, and χ belong to $\Lambda_q^p(V, g)$, $\Lambda_q^r(V, g)$, and $\Lambda_q^s(V, g)$, respectively.

4. q-INNER DERIVATIVE

Our next subject is the q-deformed inner derivative, which we will call the q-inner derivative from now on. Note that the word derivative refers to a purely algebraic property, the q-deformed Leibniz rule. For any vector field X of the underlying vector space V there is one inner derivative i_X, which acts on q-deformed forms lowering their degree by one unit,

$$i_X: \Lambda_q^p V \to \Lambda_q^{p-1} V$$

$$\phi \to i_X\phi, \qquad X \in V$$

$$(i_X\phi)(X_1, \ldots, X_{p-1}) = \phi(X, X_1, \ldots, X_{p-1}) \tag{26}$$

where $\Lambda_q^p V$ means the set of all q-deformed p-forms over V. In particular, if ϕ is a q-deformed 1-form, then we have

$$\begin{aligned}i_X\phi &= i_X(\phi_a dx^a) \\ &= \phi_a dx^a(X) \\ &= \phi_a X^a \\ &= \phi(X)\end{aligned} \tag{27}$$

where we used the summation convention over the repeated indices and

$$dx^a(X) = X^a$$

If ϕ is a q-deformed 2-form, then we find

$$
\begin{aligned}
i_X\phi(Y) &= \sum_{a<b} \phi_{ab} dx^a \wedge_q dx^b(X, Y) \\
&= \sum_{a<b} \phi_{ab}\{dx^a(X)dx^b(Y) - q^{-2}dx^a(Y)dx^b(X)\} \\
&= \sum_{a<b} X^a\phi_{ab}dx^b(Y) - q^{-2}\sum_{a<b} X^b\phi_{ab}dx^a(Y) \\
&= \sum_{a,b} X^a\phi_{ab}dx^b(Y) \qquad (28)
\end{aligned}
$$

where we used

$$\phi_{ab} = -q^{-2}\phi_{ba} \qquad (a > b) \qquad (29)$$

This can be generalized to the case of the q-deformed p-form ϕ; then we have

$$
\begin{aligned}
&i_X\phi(X_1, \ldots, X_{p-1}) \\
&= \sum_{i_1, i_2, \ldots, i_{p-1}} \left(\sum_j X^j \phi_{j i_1 \cdots i_{p-1}}\right) dx^{i_1}(X_1) \wedge_q \cdots \wedge_q dx^{i_{p-1}}(X_{p-1}) \qquad (30)
\end{aligned}
$$

Immediate properties of the q-inner derivative are as follows.

(a) i_X is a linear mapping.
(b) i_X is linear in X,

$$i_{X+Y} = i_X + i_Y$$

$$i_{aX} = a i_X, \qquad a \in R$$

(c) The q-deformed Leibniz rule is

$$i_X(\phi \wedge_q \psi) = i_X\phi \wedge_q \psi + (-q^{-2})^p \phi \wedge_q i_X\psi, \qquad \phi \in \Lambda_q^p V_q$$

(d) We have

$$i_{X_1} i_{X_2} = -q i_{X_2} i_{X_1}$$

(e) Finally,

$$i_X^2 = 0$$

We will check the property (c) for the case that $\phi \in \Lambda_q^2$ and $\psi \in \Lambda_q^1$, where

$$\phi = \sum_{a<b} \phi_{ab} dx^a \wedge_q dx^b, \qquad \phi_{ab} = -q^{-2}\phi_{ba}$$

$$\psi = \sum_c \psi_c dx^c$$

Then we find

$$
\begin{aligned}
& i_X(\phi \wedge_q \psi)(Y, Z) \\
&\quad = \sum_{a<b} \sum_c \phi_{ab}\psi_c dx^a \wedge_q dx^b \wedge_q dx^c(X, Y, Z) \\
&\quad = \sum_{a<b} \sum_c \phi_{ab}\psi_c[dx^a(X)\, dx^b(Y)\, dx^c(Z) \\
&\qquad - q^{-2}dx^a(Y)\, dx^b(X)\, dx^c(Z) \\
&\qquad - q^{-2}dx^a(X)\, dx^b(Z)\, dx^c(Y) + q^{-4}dx^a(Y)\, dx^b(Z)\, dx^c(X) \\
&\qquad + q^{-4}dx^a(Z)\, dx^b(X)\, dx^c(Y) \\
&\qquad - q^{-6}dx^a(Z)\, dx^b(Y)\, dx^c(X)], \qquad X, Y, Z \in V
\end{aligned}
$$

Here the first and second terms in right-hand side of the above equation give

$$i_X\phi(Y) \wedge_q \psi(Z)$$

the third and fifth terms give

$$-q^{-2}i_X\phi(Z) \wedge_q \psi(Y)$$

so adding two results leads to

$$i_X\phi \wedge_q \psi(Y, Z)$$

Similarly, the fourth and sixth terms give

$$(-q^{-2})^2\phi \wedge_q i_X\psi(Y, Z)$$

The general proof of (c) is given in the Appendix.

Now we prove property (d); we have

$$
\begin{aligned}
(i_{X_1}i_{X_2}\phi)(X_3, \ldots, X_p) &= (i_{X_2}\phi)(X_1, X_3, \ldots, X_p) \\
&= \phi(X_2, X_1, X_3, \ldots, X_p)
\end{aligned}
$$

and

$$
\begin{aligned}
(i_{X_2}i_{X_1}\phi)(X_3, \ldots, X_p) &= (i_{X_1}\phi)(X_2, X_3, \ldots, X_p) \\
&= \phi(X_1, X_2, X_3, \ldots, X_p)
\end{aligned}
$$

Since $\phi \in \Lambda_q^p$ is q-alternating, we have

$$(i_{X_1}i_{X_2}\phi) = -q(i_{X_2}i_{X_1}\phi)$$

Property (e) is easily proved from property (d).

5. q-DEFORMED SYMPLECTIC MANIFOLD AND q-DEFORMED HAMILTON EQUATION

In this section we discuss the q-deformed symplectic manifold and a quantization of a q-deformed classical Hamiltonian system. We start with the q-deformed symplectic two-form

$$\Omega = d\theta = dp \wedge_q dx \qquad d\Omega = 0 \tag{31}$$

where θ is called a q-deformed canonical 1-form and is given by

$$\theta = p\,dx \tag{32}$$

Here (x, p) means the local coordinate of the q-deformed symplectic manifold M_q. Let TM_q and T^*M_q be the q-deformed tangent bundle and q-deformed cotangent bundle, respectively. Then for an arbitrary element of TM_q given by

$$X = a\frac{\partial}{\partial p} + b\frac{\partial}{\partial x} \tag{33}$$

it holds that

$$i_X\Omega = -q^{-2}b\,dp + a\,dx \tag{34}$$

Proof. The proof follows from the definition of the q-inner derivative i_X. Let $\Omega = dp \wedge_q dx$; then

$$\begin{aligned} i_X\Omega &= i_X(dp \wedge_q dx) \\ &= i_X dp \wedge_q dx + (-q^{-2})dp \wedge_q i_X dx \\ &= X(p)\,dx - q^{-2}X(x)\,dp \\ &= a\,dx - q^{-2}b\,dp \end{aligned} \tag{35}$$

Similarly, for an arbitrary element of T^*M_q given by

$$\omega = f\,dp + g\,dx \tag{36}$$

we have

$$\Omega^{-1}(f\,dp + g\,dx) = g\frac{\partial}{\partial p} - q^2 f\frac{\partial}{\partial x} \tag{37}$$

Proof. From (34), we have

$$i_X\Omega = dx \qquad \text{for} \quad X = \frac{\partial}{\partial p}$$

$$i_X\Omega = -q^{-2}dp \qquad \text{for} \quad X = \frac{\partial}{\partial x} \tag{38}$$

Using these, we have

$$\Omega^{-1}(f\,dp + g\,dx) = f\Omega^{-1}(dp) + g\Omega^{-1}(dx)$$
$$= -q^2 f \frac{\partial}{\partial x} + g \frac{\partial}{\partial p} \tag{39}$$

Thus we reach the following theorem.

Theorem. The fundamental symplectic q-deformed 2-form Ω on the q-deformed cotangent bundle $P_q = T^*M_q$ generates the Hamilton equation of motion

$$\dot{p} = -\frac{\partial H}{\partial x}, \qquad \dot{x} = q^2 \frac{\partial H}{\partial p} \tag{40}$$

Proof. The differential of the Hamiltonian $H(x, p)$ is a q-deformed 1-form on P_q,

$$dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial p} dp \tag{41}$$

By (39) we have

$$\Omega^{-1} df = \frac{\partial f}{\partial x} \frac{\partial}{\partial p} - q^2 \frac{\partial f}{\partial p} \frac{\partial}{\partial x} \tag{42}$$

Hence

$$\Omega^{-1} dH = \frac{\partial H}{\partial x} \frac{\partial}{\partial p} - q^2 \frac{\partial H}{\partial p} \frac{\partial}{\partial x} = -X \tag{43}$$

The vector field $X = -\Omega^{-1}dH$ determines a system of differential equations on P_q. Finally we have the q-deformed Hamilton equations of motion

$$\dot{p} = -\frac{\partial H}{\partial x}$$
$$\dot{x} = q^2 \frac{\partial H}{\partial p} \tag{44}$$

Therefore we infer the analytic expression for the q-deformed Poisson bracket on the set of observables on P_q to be induced by the q-deformed symplectic structure on T^*M_q

$$(\Omega^{-1}df)g = \{f, g\}_q$$
$$= \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - q^2 \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \tag{45}$$

Equation (45) means that

$$\{f, g\}_q = -q^2\{g, f\}_{q^{-1}} \tag{46}$$

6. CONCLUSIONS

In this paper we have newly defined the q-deformed differential forms and quantum-algebra-valued q-deformed forms. In the latter case, we adopt the Cartesian version of the quantum algebra. Using these definitions, we have obtained a q-deformed inner derivative and discussed its properties. As a physical application, we have discussed the q-deformed Hamilton equation and q-deformed Poisson bracket, where we use the properties of q-deformed inner derivative. We think that much will be accomplished in this direction. In particular we hope that the q-deformed Lagrangian equation of motion of the q-deformed mechanics will be constructed in the near future.

APPENDIX

In this appendix we prove that for $\phi \in \Lambda_q^p$ and $\psi \in \Lambda_q^l$,

$$i_X(\phi \wedge_q \psi) = i_X\phi \wedge_q \psi + (-q^{-2})^p \phi \wedge_q i_X\psi$$

We have

$$i_X(\phi \wedge_q \psi)(X_2, \ldots, X_{p+l})$$
$$= (\phi \wedge_q \psi)(X_1, X_2, \ldots, X_{p+l})$$
$$= \sum_{i_1<\cdots<i_p} \sum_{i_{p+1}<\cdots<i_{p+l}} \phi_{i_1\cdots i_p}\psi_{i_{p+1}\cdots i_{p+l}}$$
$$\times dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_{p+l}}(X_1, \ldots, X_{p+l})$$
$$= \sum_{i_1<\cdots<i_p} \sum_{i_{p+1}<\cdots<i_{p+l}} \phi_{i_1\cdots i_p}\psi_{i_{p+1}\cdots i_{p+l}}$$
$$\times \sum_{\sigma\in S_{p+l}} sgn_q\sigma\, dx^{i_1}(X_{\sigma(1)}) \cdots dx^{i_{p+l}}(X_{\sigma(p+l)})$$
$$= \sum_{i_1<\cdots<i_p} \sum_{i_{p+1}<\cdots<i_{p+l}} \phi_{i_1\cdots i_p}\psi_{i_{p+1}\cdots i_{p+l}}$$
$$\times \sum_{\sigma'\in S_{p+l},\sigma'(1)=1,\sigma'(i)=\sigma(i),i\neq 1} sgn_q\sigma'$$

$$
\begin{aligned}
&\times dx^{i_1}(X_1)\, dx^{i_2}(X_{\sigma'(2)}) \cdots dx^{i_p}(X_{\sigma'(p)}) \\
&\times dx^{i_{p+1}}(X_{\sigma'(p+1)}) \cdots dx^{i_{p+l}}(X_{\sigma'(p+l)}) \\
&+ (-q^{-2})^p \sum_{i_1<\cdots<i_p} \sum_{i_{p+1}<\cdots<i_{p+l}} \phi_{i_1\cdots i_p}\psi_{i_{p+1}\cdots i_{p+l}} \\
&\times \sum_{\sigma''\in S_{p+l},\sigma''(p+1)=1,\sigma''(i)=\sigma(i),i\neq p+1} sgn_q\sigma'' \\
&\times dx^{i_1}(X_{\sigma''(1)}) \cdots dx^{i_p}(X_{\sigma''(p)}) \\
&\times dx^{i_{p+1}}(X_1) dx^{i_{p+2}}(X_{\sigma''(p+2)}) \cdots dx^{i_{p+l}}(X_{\sigma'(p+l)})
\end{aligned}
$$

where we used the following properties:

$$sgn_q\sigma' = sgn_q\sigma$$

and

$$
\begin{aligned}
sgn_q\delta'' &= q^{-R(\sigma''(1)\cdots\sigma''(p+l))}E^{1\cdots p+l}_{\sigma''(1)\cdots\sigma''(p+l)} \\
&= q^{-R(\sigma(1),\cdots\sigma(p),\sigma(p+1)=1,\sigma(p+2),\cdots\sigma(p+l))}E^{1\cdots p+l}_{\sigma(1),\dots,\sigma(p),\sigma(p+1)=1,\sigma(p+2),\dots,\sigma(p+l)} \\
&= (-q^{-2})^p q^{-R(\sigma(1),\dots,\sigma(p),\sigma(p+2),\dots,\sigma(p+l))}E^{1,\dots,p,p+2,\dots,p+l}_{\sigma(1),\dots,\sigma(p),\sigma(p+2),\dots,\sigma(p+l)}
\end{aligned}
$$

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